Recitation 27 - Homework 3

Question 1

A monopolist produces a good at constant marginal cost $c > 0$, and sells an amount $q \geq 0$ to a consumer. The consumer's utility is

$$u(q, t, \theta) = \theta v(q) - t$$

where $\theta \in \{\underline{\theta}, \bar{\theta}\}$ with $\bar{\theta} > \underline{\theta} > 0, v' > 0, v'' < 0$ and $v(0) = 0$. The monopolist believes that the consumer's type is $\underline{\theta}$ with probability $\underline{p}$, and $\bar{\theta}$ with probability $\bar{p}$ where $\underline{p} + \bar{p} = 1$. The consumer has an outside option of 0. The monopolist designs a menu of (q, t) bundles, where q is the amount purchased and t is the purchase price, for the consumer subject to individual rationality and incentive compatibility.

(a) Show that if the low type's participation constraint is satisfied then the high type's participation constraint is satisfied, and that, at an optimum, the low type's participation constraint binds.

(b) Given a bundle for the low type, draw a picture in (q, t) -space for the set of feasible bundles for the high type. Show that the high type's incentive compatibility constraint binds at an optimum.

(c) Derive the monopolist's profit-maximizing menu of bundles. Hint: you may want to distinguish between $\bar{p}\bar{\theta} < \underline{\theta}$ and $\bar{p}\bar{\theta} \geq \underline{\theta}$

(d) Rewrite each contract in the optimal menu as a two-part tariff: a fixed fee together with a marginal price per unit, instead of a quantity together with a total price for that quantity.

Solution

a) For any feasible(participation and incentive constraints hold) allocation, $q(\underline{\theta}), q(\bar{\theta}), t(\underline{\theta}), t(\bar{\theta})$ satisfies

$$\underline{\theta} v(q(\underline{\theta})) - t(\underline{\theta}) \geq 0 (1) \text{ and}$$
$$\bar{\theta} v(q(\bar{\theta})) - t(\bar{\theta}) \geq \bar{\theta} v(q(\underline{\theta})) - t(\underline{\theta}) (2)$$

(2) $\Rightarrow \bar{\theta} v(q(\bar{\theta})) - t(\bar{\theta}) \geq \bar{\theta} v(q(\underline{\theta})) - t(\underline{\theta}) > \underline{\theta} v(q(\underline{\theta})) - t(\underline{\theta}) \geq 0$ In last line, second last inequality follows from $\bar{\theta} > \underline{\theta}$.

Hence, if participation constraint(PC) for low type and incentive compatibility(IC) constraint for high type are satisfied, so is participation constraint for high type. To show that PC for low type binds, suppose not.

Then, both $t(\underline{\theta})$ and $t(\bar{\theta})$ can be increased a little by same amount c. Then both PCs still hold as they were earlier holding with strict inequality.

Moreover, ICs also hold since both transfers changed by same amount. Hence, increasing transfer increases profit for monopolist meaning earlier allocation was not optimal. Hence, optimal allocation has

PC for low-type binding with equality.

b) Given $q(\underline{\theta}), t(\underline{\theta}), q(\bar{\theta}), t(\bar{\theta})$ satisfy IC. We don't care about PC as we showed in part a, PC holding for low-type ensures PC holding for high-type.

$$\bar{\theta}v(q(\bar{\theta})) - t(\bar{\theta}) \geq \bar{\theta}v(q(\underline{\theta})) - t(\underline{\theta}) = c$$

$v(q(\underline{\theta})) - t(\underline{\theta}) = c$ is crossing only once $q = 0$

Moreover, note that optimal will be on the curve. Otherwise, monopolist can increase $t(\bar{\theta})$ slightly and increase her profit. In that case, he is still below the IC curve. IC for low-type will also hold since lying has lower payoff now as $t(\bar{\theta})$ increased. Hence, optimal is on curve and IC for high-type binds.

c)Monpolist problem is

$$\max_{t,t,\bar{q},q} \underline{p}(\underline{t} - c\underline{q}) + \bar{p}(\bar{t} - c\bar{q}) \text{ subject to}$$
$$\bar{\theta}v(\bar{q}) - \bar{t} = \bar{\theta}v(\underline{q}) - \underline{t} (\text{IC for high-type })$$
$$\underline{\theta}v(\underline{q}) = \underline{t} (\text{PC for low-type })$$

We ignore IC for low-type in problem and later verify that holds. This is not a concave programming problem. But it can be converted into one by writing q' s as a convex function of utility. Doing that and solving for first-order gives

$$v'(\bar{q}) = \frac{c}{\bar{\theta}}$$
$$v'(\underline{q})(\underline{\theta} - \bar{p}\bar{\theta}) = \underline{p}c$$

When $\underline{\theta} - \bar{p}\bar{\theta} \leq 0, \underline{q} = 0$ and monopolist sells just to high-type(this corresponds to when high-type are too many, that is, high $\bar{p}$ or their valuation $\bar{\theta}$ is too high. So, seller ignores low-type and focuses on high-type.). Hence, optimal quantity for monopolist is given by

$$\bar{q} = v^{-1}\left(\frac{c}{\bar{\theta}}\right)$$

$$\underline{q} = \begin{cases} 0, \text{ if } \underline{\theta} \leq \bar{p}\bar{\theta} \\ v'^{-1}\left(\frac{\underline{p}c}{\underline{\theta} - \bar{p}\bar{\theta}}\right), \text{ otherwise} \end{cases}$$

Note that $\underline{q} < \bar{q}$. When $\underline{q} = 0$, this holds trivially. In other case of $\underline{q} > 0$, by concavity of v

$$\underline{q} < \bar{q} \Leftrightarrow v'(\underline{q}) > v'(\bar{q}) \Leftrightarrow \frac{c}{\bar{\theta}} < \frac{\underline{p}c}{\underline{\theta} - \bar{p}\bar{\theta}} \Leftrightarrow \underline{\theta} < \bar{\theta}$$

Transfers are then given by IC and PC constraints in the problem.

$$\underline{\theta}v(\underline{q}) = \underline{t}$$
$$\bar{\theta}v(\bar{q}) - \bar{t} = \bar{\theta}v(\underline{q}) - \underline{t}$$

Only thing needed is to verify IC for low type holds, that is, $\underline{\theta}v(\bar{q}) - \bar{t} \leq \underline{\theta}v(q) - \underline{t} = 0$. Proof is as follows.

$$\underline{\theta}v(\bar{q}) - \bar{t} = \underline{\theta}v(\bar{q}) - \underline{t} + \underline{t} - \bar{t}$$

$= \underline{\theta}v(\bar{q}) - \underline{t} + \bar{\theta}v(\bar{q}) - \bar{\theta}v(\bar{q})$ (Substituting $\underline{t} - \bar{t}$ from IC of high-type) $= \underline{\theta}v(\bar{q}) - \underline{\theta}v(\underline{q}) + \bar{\theta}v(\bar{q}) - \bar{\theta}v(\bar{q})$ (Substituting $\underline{t}$ from PC of low-type)

$$= (\underline{\theta} - \bar{\theta})(v(\bar{q}) - v(\underline{q})) < 0$$

Last step follows from $\underline{\theta} < \bar{\theta}, \underline{q} < \bar{q}$ and v is increasing.

d) Total amount paid by low and high types are $\underline{t}$ and $\bar{t}$ respectively. Let the fixed fees and per unit fees for low and high types be (FF_{low}, PPU_{low}) and $(FF_{\text{high}}, PPU_{\text{high}})$ respectively. We first characterise the solution and later show that each type chooses the bundle corresponding to her type. Low type chooses (FF_{low}, PPU_{low}) and buys q and high type chooses FF_{high}, PPU_{high} and buys $\bar{q}$. Then first-order conditions of each type gives $v'(\underline{q}) = PPU_{low} = \left(\frac{pc}{\underline{\theta}-\bar{p}\theta}\right)$ and $v'(\bar{q}) = PPU_{high} = \frac{c}{\theta}$ gives us the per unit prices of both types. Then fixed fees can be obtained from following equations.

$$FF_{low} + \underline{q}PPU_{low} = \underline{t}$$
$$FF_{high} + \bar{q}PPU_{high} = \bar{t}$$

Now, we just need to show each type chooses the bundle meant for her. For simplicity, we rewrite q as q_{low} and $\bar{q}$ as q_{high}. Similarly, we rewrite $\underline{t}$ as t_{low} and $\bar{t}$ as t_{high}. For $i \in \{\text{low, high}\}$, if type i chooses bundle meant for j, her objective is

$$\max_q \theta_i v(q) - FF_j - v'(q_j)\, q$$

objective is

$$\max_q \theta_i v(q) - FF_j - v'(q_j)\, q$$

Say q^* solves this. Then, her payoff from choosing wrong type is

$$\begin{aligned}\theta_i v(q^*) - FF_j - v'(q_j)\, q^* &= \theta_i v(q^*) - FF_j - v'(q_j)\, q_j + v'(q_j)(q_j - q^*) \\ &= \theta_i v(q^*) - t_j + v'(q_j)(q_j - q^*) \text{ (by construction of } t_j) \\ &= \theta_i v(q_j) - t_j + v'(q_j)(q_j - q^*) + \theta_i (v(q^*) - v(q_j))\end{aligned}$$

$\leq \theta_i v(q_i) - t_i + v'(q_j)(q_j - q^*) + \theta_i (v(q^*) - v(q_j))$ (by incentive compatibility of i) If we show the term $v'(q_j)(q_j - q^*) + \theta_i (v(q^*) - v(q_j))$ to be negative, we are done as then choosing the other bundle j gives lower objective than choosing the bundle meant for i. By continuity and differentiability of v, there exists $\hat{q}$ between q^* and q_j satisfying $v'(\hat{q})(q^* - q_j) = v(q^*) - v(q_j)$. Then

$$v'(q_j)(q_j - q^*) + \theta_i (v(q^*) - v(q_j)) = (\theta_i v'(\hat{q}) - v'(q_j))(q^* - q_j)$$

Now note that high type always chooses more amount than low type for any price as her marginal utility is lower than that of low-type. SO, if i is high and j is low, $q^* > \hat{q} > q_j$ and $v'(\hat{q}) < v'(q_j)$. So, the term $(\theta_i v'(\hat{q}) - v'(q_j))(q^* - q_j)$ is negative as first term is negative and second one is positive. On the other hand, if i is low and j is high, $q^* < \hat{q} < q_j$ and $v'(\hat{q}) > v'(q_j)$. In this case also, $(\theta_i v'(\hat{q}) - v'(q_j))(q^* - q_j)$ is negative and we get

$$\theta_i v(q_j) - t_j \leq \theta_i v(q_i) - t_i$$

Thus everyone chooses out of 2 bundles the bundle meant for her.

Question 2

Consider a single consumer with utility function $U(x, T|\theta) = v(x, \theta) - T$, where x is the amount consumed of a good, T is the amount of money paid for the good, and θ is a type drawn according to the CDF F with support $[\underline{\theta}, \bar{\theta}]$, where $0 < \underline{\theta} < \bar{\theta} < \infty$ and F has a continuous PDF f such that $f(\theta) > 0$ for all θ in the support of F. Assume that v is twice continuously differentiable and has the single crossing property:

$$\frac{\partial^2 v(x, \theta)}{\partial x \partial \theta} > 0 \quad \forall (x, \theta)$$

A monopolist with marginal cost of producing the good $c \geq 0$ considers designing a mechanism for the consumer. A mechanism is a pair (x, t), where $x(\theta)$ denotes the quantity of the good consumed by the agent and $t(\theta)$ the amount paid to the monopolist if he reports type θ. A feasible mechanism for the monopolist must be incentive compatible and individually rational, where the consumer's outside option is normalized to zero for every type.

(a) Assume that the mechanism (x, t) is incentive compatible. i. Show that x must be weakly increasing, i.e., $x(\theta) \leq x(\theta')$ whenever $\theta < \theta'$ ii. Show that $U'(\theta) = \partial v(q(\theta), \theta)/\partial \theta$ at almost every θ in the support of F

$$\text{where } U(\theta) = v(x(\theta), \theta) - t(\theta)$$

iii. Conclude that

$$U(\theta) = \underline{U} + \int_{\underline{\theta}}^{\theta} \frac{\partial v(q(\tau), \tau)}{\partial \theta} d\tau$$

(b) Show that the mechanism (x, t) is incentive compatible if q is weakly increasing and ($*$) above holds.

(c) Let $x^*(\theta)$ be the pointwise (in θ) solution of the equation

$$\frac{\partial v(x(\theta), \theta)}{\partial x} - \frac{1 - F(\theta)}{f(\theta)} \frac{\partial^2 v(x(\theta), \theta)}{\partial x \partial \theta} = c$$

Show that, if x^* is weakly increasing then it maximizes the monopolist's profit amongst all incentive compatible, individually rational mechanisms.

(d) Show that, at the optimum x^*, there are no distortions at the top, i.e., the highest type's marginal benefit from consuming the good equals the marginal cost of producing the good, and that the lowest type's individual rationality constraint binds.

Solution

a) i) We need to show $x(\theta) < x(\theta')$ whenever $\theta < \theta'$. For any θ, θ' with $\theta' > \theta$, IC gives

$$v(x(\theta), \theta) - t(\theta) \geq v(x(\theta'), \theta) - t(\theta')$$
$$v(x(\theta'), \theta') - t(\theta') \geq v(x(\theta), \theta') - t(\theta)$$

Adding both gives

$$v(x(\theta), \theta') - v(x(\theta), \theta) \leq v(x(\theta'), \theta') - v(x(\theta'), \theta)$$

$$\Rightarrow v(x(\theta), \theta)+\int_{\theta}^{\theta^{\prime}} \frac{\partial v(x(\theta), \tau)}{\partial \tau} d \tau \leq v\left(x\left(\theta^{\prime}\right), \theta\right)+\int_{\theta}^{\theta^{\prime}} \frac{\partial v\left(x\left(\theta^{\prime}\right), \tau\right)}{\partial \tau} d \tau(*)$$

Now suppose $x(\theta) \geq x\left(\theta^{\prime}\right)$. Then first term of left side is greater than first term of right side as v is increasing. Moreover, $\frac{\partial v(x(\theta), \tau)}{\partial \tau}>\frac{\partial v\left(x\left(\theta^{\prime}\right), \tau\right)}{\partial \tau}$ by single crossing property. So, second term of left side is greater than second term of right side. Thus left side is greater than right side contradicting *. Hence, $x(\theta) \geq x\left(\theta^{\prime}\right)$ is not possible. So, $x(\theta)<x\left(\theta^{\prime}\right)$ whenever $\theta<\theta^{\prime}$

a) ii) By incentive constraint, for any $\theta, \theta^{\prime}$

$$t\left(\theta^{\prime}\right)-t(\theta) \geq v\left(x\left(\theta^{\prime}\right), \theta\right)-v(x(\theta), \theta)$$

Setting $\theta^{\prime}=\theta+\epsilon$

$$\frac{t(\theta+\epsilon)-t(\theta)}{\epsilon} \geq \frac{v(x(\theta+\epsilon), \theta)-v(x(\theta), \theta)}{\epsilon}$$

$$\Rightarrow \frac{t(\theta+\epsilon)-t(\theta)}{\epsilon} \geq\left(\frac{v(x(\theta+\epsilon), \theta)-v(x(\theta), \theta)}{x(\theta+\epsilon)-x(\theta)}\right)\left(\frac{x(\theta+\epsilon)-x(\theta)}{\epsilon}\right)$$

Taking limit on $\epsilon \rightarrow 0$

$$t^{\prime}(\theta) \geq \frac{\partial v(x(\theta), \theta)}{\partial x(\theta)} \frac{\partial x(\theta)}{\partial \theta}$$

Similarly,

$$t(\theta)-t\left(\theta^{\prime}\right) \geq v\left(x(\theta), \theta^{\prime}\right)-v\left(x\left(\theta^{\prime}\right), \theta^{\prime}\right)$$

Setting $\theta^{\prime}=\theta+\epsilon$

$$\frac{t(\theta)-t(\theta+\epsilon)}{\epsilon} \geq \frac{v(x(\theta),(\theta+\epsilon))-v(x(\theta+\epsilon),(\theta+\epsilon))}{\epsilon}$$

$$\Rightarrow \frac{v(x(\theta+\epsilon),(\theta+\epsilon))-v(x(\theta),(\theta+\epsilon))}{\epsilon} \geq \frac{t(\theta+\epsilon)-t(\theta)}{\epsilon}$$

Taking limit on $\epsilon \rightarrow 0$

$$\frac{\partial v(x(\theta), \theta)}{\partial x(\theta)} \frac{\partial x(\theta)}{\partial \theta} \geq t^{\prime}(\theta)$$

From (1) and (2)

$$\frac{\partial v(x(\theta), \theta)}{\partial x(\theta)} \frac{\partial x(\theta)}{\partial \theta}=t^{\prime}(\theta)$$

$$\Rightarrow U^{\prime}(\theta)=\frac{\partial v(x(\theta), \theta)}{\partial \theta}+\frac{\partial v(x(\theta), \theta)}{\partial x(\theta)} \frac{\partial x(\theta)}{\partial \theta}-t^{\prime}(\theta)=\frac{\partial v(x(\theta), \theta)}{\partial \theta}(3)$$

a) iii)

$$U(\theta)=U(\underline{\theta})+\int_{\underline{\theta}}^{\theta} U^{\prime}(\tau) d \tau$$

$$=U(\underline{\theta})+\int_{\underline{\theta}}^{\theta} \frac{\partial v(x(\tau), \tau)}{\partial \tau} d \tau(\text { From } 3)$$

b) We need to show for any θ, θ' $v(x(\theta), \theta) - t(\theta) \geq v(x(\theta'), \theta) - t(\theta')$ if condition in a(iii) holds Note that condition in a(iii) implies $U'(\theta) = \frac{\partial v(x(\theta),\theta)}{\partial \theta}$. Then

$$\begin{gathered} v(x(\theta), \theta) - t(\theta) \geq v(x(\theta'), \theta) - t(\theta') \\ \Leftrightarrow U(\theta) \geq U(\theta') + v(x(\theta'), \theta) - v(x(\theta'), \theta') \\ \Leftrightarrow U(\theta) - U(\theta') \geq v(x(\theta'), \theta) - v(x(\theta'), \theta') \\ \Leftrightarrow \int_{\theta'}^{\theta} U'(\tau) d\tau \geq \int_{\theta'}^{\theta} \frac{\partial v(x(\theta'),\tau)}{\partial \tau} d\tau \\ \Leftrightarrow \int_{\theta'}^{\theta} \frac{\partial v(x(\tau),\tau)}{\partial \tau} d\tau \geq \int_{\theta'}^{\theta} \frac{\partial v(x(\theta'),\tau)}{\partial \tau} d\tau \\ \Leftrightarrow \int_{\theta'}^{\theta} \left(\frac{\partial v(x(\tau),\tau)}{\partial \tau} - \frac{\partial v(x(\theta'),\tau)}{\partial \tau} \right) d\tau \geq 0 \end{gathered}$$

If $\theta' < \theta$, term under integral is positive as $x(\tau) > x(\theta')$ by monotonicity of x and by single-crossing, first term is greater than second term. Hence, above inequality holds. A similar argument works for $\theta' > \theta$. Then term under integral is negative and the integral is positive as lower limit is greater than upper limit. Hence, we have proved above condition holds and mechanism is incentive compatible.

c) Incentive compatibility of any mechanism means

$$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \frac{\partial v(x(\tau), \tau)}{\partial \tau} d\tau$$

Hence, monopolist maximizing profit subject to incentive compatibility can be written as

$$\begin{gathered} \max_{x(\theta),t(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} (t(\theta) - c(x(\theta))) dF(\theta) \text{ subject to} \\ v(x(\theta), \theta) - t(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \frac{\partial v(x(\tau),\tau)}{\partial \tau} d\tau \forall \theta \end{gathered}$$

since objective is linear and constraint holds with equality. Above problem can be converted to a concave programming problem by changing equality to $\geq$ Let multiplier for constraint at θ be $\lambda(\theta)$. First-order condition with respect to $t(\theta)$ is

$$dF(\theta) = \lambda(\theta)$$

First-order condition with respect to $x(\theta)$ is

$$-cdF(\theta) + \lambda(\theta) \frac{\partial v(x(\theta), \theta)}{\partial \theta} - \int_{\theta}^{\bar{\theta}} \frac{\partial^2 v(x(\theta), \theta)}{\partial \theta \partial x} \lambda(\tau) = 0$$

Substituting λ from first FOC into second,

$$\begin{gathered} -c\lambda(\theta) + \lambda(\theta) \frac{\partial v(x(\theta),\theta)}{\partial \theta} - \int_{\theta}^{\bar{\theta}} \frac{\partial^2 v(x(\theta),\theta)}{\partial \theta \partial x} dF(\tau) = 0 \\ \Rightarrow cf(\theta) = f(\theta) \frac{\partial v(x(\theta), \theta)}{\partial \theta} - (1 - F(\theta)) \frac{\partial^2 v(x(\theta), \theta)}{\partial \theta \partial x} \\ c = \frac{\partial v(x(\theta), \theta)}{\partial \theta} - \frac{1 - F(\theta)}{f(\theta)} \frac{\partial^2 v(x(\theta), \theta)}{\partial \theta \partial x} \end{gathered}$$

So, best incentive compatible mechanism should satisfy this equation. Hence, if x is weakly increasing then it maximizes the monopolist's profit amongst all incentive compatible, individually rational mechanisms.

d) At upper limit $\bar{\theta}, 1 - F(\bar{\theta}) = 0$ and $c = \frac{\partial v(x(\theta),\theta)}{\theta \theta_0}$. Thus, highest type's marginal benefit from consuming the good equals the marginal cost of producing the good. Moreover, $U(\underline{\theta})$ should be 0 otherwise $U(\underline{\theta}) > 0$ implies monopolist can increase all $t(\theta)$ by some small amount and increase its profit. Participation constraint will not be violated as all $U(\theta)$ are positive and incentive constraint is not violated since all transfers change by same amount.

Question 3

A monopolist faces a single consumer with utility function $u = \theta q - \frac{1}{2}q^2 - T$, where θ is private information of the consumer, q is the level of consumption and T is the amount of money that the consumer pays the monopolist. The monopolist's cost of producing q equals $\frac{1}{2}cq^2$ for some constant $c > 0$. The consumer's reservation utility equals 0. The (Pareto) CDF of θ is $F(\theta) = 1 - \theta^{-\alpha}$ for all $\theta \in [1, \infty)$, where $\alpha > 1$

(a) Derive the monopolist's optimal bundle (q, T) assuming that it knows θ.

(b) Write down the monopolist's nonlinear pricing problem.

(c) Derive the optimal nonlinear pricing schedule and compare it with (a).

(d) What can you say about distortions at the top? (Here, "top" means $\theta \to \infty$.)

Solution

a) Knowing θ, monopolist extracts all surplus ($T = \theta q - 0.5q^2$) and problem is to maximize $T - 0.5cq^2 = \theta q - 0.5q^2 - 0.5cq^2$ First order condition gives

$$q = \frac{\theta}{1+c}, T = \frac{\theta^2(1+2c)}{2(1+c)^2}$$

b) Monopolist solves

$$\max_{T(\theta),q(\theta)} \int_\theta \left(T(\theta) - \frac{1}{2}c(q(\theta))^2\right) dF(\theta) \text{ subject to}$$
$$\theta q(\theta) - \frac{1}{2}(q(\theta))^2 - T(\theta) \geq 0 \forall \theta (\text{PC})$$
$$\theta q(\theta) - \frac{1}{2}(q(\theta))^2 - T(\theta) \geq \theta q(\theta') - \frac{1}{2}\left(q(\theta')\right)^2 - T(\theta') \forall \theta, \theta' (\text{IC})$$

c) Note that utility satisfies single-crossing property, i.e., $\frac{\partial^2 u}{\partial \theta \partial} = 1 > 0$. So, similar to question2 part a, any incentive compatible allocation satisfies

$$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(\tau) d\tau$$
$$\Rightarrow \theta q(\theta) - \frac{1}{2}(q(\theta))^2 - T(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(\tau) d\tau$$

Maximizing the objective subject to this constraint gives first order conditions. Let multiplier on constraint for θ be $\lambda(\theta)$. Then FOCs are

$$dF(\theta) = \lambda(\theta)$$
$$-cq(\theta) dF(\theta) + \lambda(\theta)(\theta - q(\theta)) + q(\theta)(1 - F(\theta)) = 0$$

These two FOCs in turn give

$$cq(\theta) = \theta - q(\theta) - \frac{1 - F(\theta)}{f(\theta)}$$

$$\Rightarrow (c+1)q(\theta) = \theta - \frac{\theta^{-\alpha}}{\alpha\theta^{-\alpha-1}} = \theta - \frac{\theta}{\alpha}$$
$$\Rightarrow q(\theta) = \frac{\theta}{c+1}\frac{\alpha-1}{\alpha}$$

Hence, $q(\theta)$ is increasing in θ verifying our guess. Moreover, $q(\theta)$ is a fraction $\frac{\alpha-1}{\alpha}$ of optimal $q(\theta)$ in part a. To get prices, we use

$$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \frac{\partial v(x(\tau),\tau)}{\partial \tau} d\tau$$
$$\Rightarrow \theta q(\theta) - \tfrac{1}{2}(q(\theta))^2 - T(\theta) = \int_{\underline{\theta}}^{\theta} q(\tau)d\tau (\text{ since } U(\underline{\theta}) = 0)$$
$$\Rightarrow \theta q(\theta) - \tfrac{1}{2}(q(\theta))^2 - T(\theta) = \tfrac{\theta^2}{2(c+1)}\tfrac{\alpha-1}{\alpha}$$

This gives $T(\theta)$

d) Even at top, distortion does not go away. Basically, the distortion is to ensure no one of higher type pretends to be a lower type. In question 2, there is no one of higher type at θ. However, in this question, there is always someone of higher type. So, there is always a distortion.

Question 5

A seller owns an object, and values it at 0. There is a buyer with valuation $v \sim U[0,1]$ The seller does not know the buyer's valuation, and designs an optimal mechanism to fulfill some objective, whereby the seller asks for the buyer's valuation and then awards the object to the buyer with probability $q(v)$ and charges the buyer an amount of money $p(v)$

(a) Assume that the seller wants to maximize own profit, $p(v)$ i. Show that the seller's virtual surplus can be written as

$$2v - 1$$

ii. Describe the seller's optimal auction.

(b) Assume instead that the seller wants to maximize a weighted average of own profit, $p(v)$ (with weight $\alpha \in [0,1]$), and consumer surplus, $v - p(v)$ (with weight $1-\alpha$)

i. Show that the seller's virtual surplus can be written as

$$(3\alpha - 1)v + 1 - 2\alpha$$

Hint. First derive the seller's virtual surplus when the buyer's valuation v has a general CDF F with PDF $f > 0$

ii. Describe the seller's optimal auction as a function of $\alpha \in [0,1]$

Solution

a) Seller's problem is to maximize her expected profit

$$\int_0^1 p(v)f(v)dv \text{ subject to}$$
$$vq(v) - p(v) = \int_0^v q(s)ds (\text{ IC for increasing q})$$

Substituting $p(v)$ from IC, objective is

$$\int_0^1 \left(vq(v) - \int_0^v q(s)ds \right) f(v)dv$$

$$= \int_0^1 vq(v)f(v)dv - \int_0^1 \left(\int_0^v q(s)ds \right) f(v)dv$$

$$= \int_0^1 vq(v)f(v)dv - \left[\left(\int_0^v q(s)ds \right) F(v) \right]_0^1 + \int_0^1 q(v)F(v)dv \text{ (Integration by parts)}$$

$$= \int_0^1 vq(v)f(v)dv - \int_0^1 q(v)dv + \int_0^1 q(v)F(v)dv$$

$$= \int_0^1 \left[v - \frac{1-F(v)}{f(v)} \right] q(v)f(v)dv$$

So, for each individual with value v, $\left[v - \frac{1-F(v)}{f(v)} \right]$ is the virtual surplus seller can extract. Substituting $f(v) = 1$ and $F(v) = v$ for uniform distribution, it becomes $2v - 1$. Then objective is maximizing $(2v-1)q(v)$ over all v. So, optimal strategy is to set probability $q(v) = 1$ when $v \geq 0.5$ and 0 otherwise. So, sell if valuation revealed is at least 0.5 and don't sell otherwise. Then $p(v) = vq(v) - \int_0^v q(s)ds$ is 0 for $v \leq 0.5$ and 0.5 otherwise.

b) For changed consumer surplus from $v - p(v)$ to $vq(v) - p(v)$ since value to customer is not v but $vq(v)$ which is expected value.

However, $q(v)$ is always 0 or 1 when solved and when it is $0, p(v)$ is 0 due to participation constraint and consumer surplus is 0 thus making $v-p(v)$ and $vq(v)-p(v)$ same. When, $q(v)$ is 1, then also both are same. $v - p(v)$ is akin to $vq(v) - p(v)$. Seller's problem is to solve the following

$$\int_0^1 (\alpha p(v) + (1-\alpha)(vq(v) - p(v))f(v)dv \text{ subject to}$$
$$vq(v) - p(v) = \int_0^v q(s)ds \text{ (IC for increasing q)}$$

Again substituting $p(v)$ from IC, objective is

$$\int_0^1 \left[\alpha \left(vq(v) - \int_0^v q(s)ds \right) + (1-\alpha) \int_0^v q(s)ds \right] f(v)dv$$

$$= \alpha \int_0^1 vq(v)f(v)dv + (1-2\alpha) \int_0^1 \left(\int_0^v q(s)ds \right) f(v)dv$$

$$= \alpha \int_0^1 vq(v)f(v)dv + (1-2\alpha) \left[\left(\int_0^v q(s)ds \right) F(v) \right]_0^1 - (1-2\alpha) \int_0^1 q(v)F(v)dv \text{ (Integration by parts)}$$

$$= (3\alpha - 1) \int_0^1 vq(v)f(v)dv + (1-2\alpha) \int_0^v q(v)dv$$

$$= \int_0^1 \left[(3\alpha - 1)v + \frac{1-2\alpha}{f(v)} \right] q(v)f(v)dv$$

Hence, virtual surplus to be extracted from someone of type v is $(3\alpha - 1)v + \frac{1-2\alpha}{\{v\}}$ which becomes $(3\alpha - 1)v + 1 - 2\alpha$ for uniform distribution.

Hence, seller will set $q(v) = 1$ if $(3\alpha - 1)v + 1 - 2\alpha$ is positive. Note that this analysis holds only when $q(v)$ is a non-decreasing function of v. A sufficient condition for this is virtual surplus as an increasing function of v (regular case) or $\alpha > \frac{1}{3}$: Then, $q(v) = 1$ and seller sells if $(3\alpha - 1)v + 1 - 2\alpha > 0$ or

$v > \frac{2\alpha-1}{3\alpha-1} \equiv \hat{v}$

Then $p(v) = 0$ for $v < \hat{v}$ and it equals $\hat{v}$ for $v \geq \hat{v}$. So far, we have solved the regular case, $\alpha > \frac{1}{3}$. For non-regular case, note that for $\alpha \leq \frac{1}{3}$, virtual surplus is decreasing and reaches its minimum value at $v = 1$. Then its value is $\alpha > 0$. So, virtual surplus is always positive and seller always sells the object with $q(v) = 1$.

This satisfies q being non-decreasing in v. Hence, for $\alpha \leq \frac{1}{3}$, the solution $q(v) = 1 \forall v \in [0, 1]$ is non-decreasing and it maximizes the objective. SO, it is the solution. Then $p(v) = vq(v) - \int_0^v q(s)ds = 0$ Hence, the optimal auction for any α is

$$\text{When } \alpha \leq \tfrac{1}{3}, (q(v), p(v)) = (1, 0) \forall v$$
$$\text{When } \alpha > \tfrac{1}{3}, (q(v), p(v)) = (0, 0) \forall v \leq \hat{v} \equiv \tfrac{2\alpha-1}{3\alpha-1}, 1 \text{ otherwise } (q(v), p(v)) = (1, \hat{v})$$

This makes sense intuitively as for small weight on own profit and high altruism, that is, low α, seller gives the good for free.

Question 6

Consider and economy with one buyer and one seller of a single indivisible object. There are two periods of time. In the each period, the buyer learns independent pieces of information. In the first period, the buyer obtains a signal $\theta \in \{L, H\}$ such that $\Pr(\theta = H) = p$ and $\Pr(\theta = L) = 1 - p$; in the second period, the buyer obtains a signal $\sigma \sim U[0, 1]$. The buyer's valuation for the good, $v_\theta(\sigma) \geq 0$, satisfies

$$v_L(\sigma) < v_H(\sigma) \quad \forall \sigma \in (0, 1)$$

Assume that $v_\theta(\sigma)$ has a bounded, strictly positive derivative at every $\sigma \in (0, 1)$, and that, for every $\sigma \in [0, 1]$ as well as every $\theta' \neq \theta$, there is a unique $\sigma' \in [0, 1]$ such that $v_\theta(\sigma) = v_{\theta'}(\sigma')$. The buyer's outside option and the seller's opportunity cost for the object are both equal to zero.

A (direct) mechanism is a pair (x, t) such that $x_\theta(\sigma) \in [0, 1]$ for every (θ, σ) is the probability that the buyer receives the object as a function of his two pieces of information, and $t_\theta(\sigma) \in \mathbb{R}$ is the money paid by the buyer to the seller. Both parties have quasi-linear utility.

(a) Assume that σ is also observed by the seller, but not θ

i. Write down the buyer's incentive compatibility and individual rationality constraints.

ii. Assume that (x^*, t^*) maximizes the seller's expected revenue. Show that both the incentive constraint for $\theta = H$ pretending to be L and the individual rationality constraint for $\theta = L$ will bind. Show that, in addition,

$$\int_0^1 [v_H(\sigma) - v_L(\sigma)] [x^*_H(\sigma) - x^*_L(\sigma)] \, d\sigma \geq 0$$

iii. Use the binding constraints to show that the seller's problem simplifies to

$$\max_x \int_0^1 [p v_H(\sigma) x_H(\sigma) + (v_L(\sigma) - p v_H(\sigma)) x_L(\sigma)] \, d\sigma \qquad \text{s.t. (1) above}$$

where the expected transfers are pinned down by the binding constraints.

iv. Find an optimal mechanism for the seller.

(b) Assume now that σ is not observed by the seller; recall the x^* you derived previously.

i. Show that for there to exist transfers that make x^* incentive compatible in this new informational regime, $x^*_\theta(\sigma)$ must be increasing in σ for each θ

ii. Suppose that $v_L(\sigma)/v_H(\sigma)$ is strictly increasing in σ. Is x^* implementable?

iii. What if x^* is strictly decreasing?

Solution

Part a.i The type L 's IC and IR are

$$v_L(\sigma)x_L(\sigma) - t_L(\sigma) \geq 0 (\text{IR})$$

$$v_L(\sigma)x_L(\sigma) - t_L(\sigma) \geq v_L(\sigma)x_H(\sigma) - t_H(\sigma) (\text{IC})$$

Type H 's constraints are

$$v_H(\sigma)x_H(\sigma) - t_H(\sigma) \geq 0 \text{ (IR)}$$
$$v_H(\sigma)x_H(\sigma) - t_H(\sigma) \geq v_H(\sigma)x_L(\sigma) - t_L(\sigma) \text{ (IC)}$$

Part a.ii First note that, if IR for L holds, so does IR for H as

$$v_H(\sigma)x_H(\sigma) - t_H(\sigma) \geq v_H(\sigma)x_L(\sigma) - t_L(\sigma) \geq v_L(\sigma)x_L(\sigma) - t_L(\sigma) \geq 0 \text{ (as } v_H(\sigma) > v_L(\sigma))$$

Moreover, if IR for L does not hold and net surplus for both types is positive, both transfer $t_L(\sigma)$ and $t_H(\sigma)$ can be increased while maintaining IR for both types. Also, IC also holds for both types as transfers are increased by same amount. For IC of H to hold with equality, suppose not. Then

$$v_H(\sigma)x_H(\sigma) - t_H(\sigma) > v_H(\sigma)x_L(\sigma) - t_L(\sigma) \geq v_L(\sigma)x_L(\sigma) - t_L(\sigma) = 0$$

In this case, $t_H(\sigma)$ can be increased while maintaining all the constraints. For H, since inequalities are strict, increasing transfer by small amount maintains them. Moreover, L has no incentive to lie as net payoff from lying is decreasing in $t_H(\sigma)$. Hence, IC for rich will hold with equality. At optimal mechanism, adding IC of H and L gives

$$v_H(\sigma)x_H(\sigma) + v_L(\sigma)x_L(\sigma) \geq v_H(\sigma)x_L(\sigma) + v_L(\sigma)x_H(\sigma)$$
$$\Leftrightarrow (v_H(\sigma) - v_L(\sigma))(x_H(\sigma) - x_L(\sigma)) \geq 0$$

Taking expectation over σ which follows uniform distribution over [0,1]

$$\int_0^1 [v_H(\sigma) - v_L(\sigma)](x_H(\sigma) - x_L(\sigma))\, d\sigma \geq 0$$

Part a.iii Seller's problem is to maximize her expected profit

$$\max_{t_\theta(\sigma), x_o(\sigma)} [pt_H(\sigma) + (1-p)t_L(\sigma)]\, d\sigma \text{ subject to}$$
$$t_L(\sigma) = v_L(\sigma)x_L(\sigma)(\text{ IR for } L \text{ binds })$$

$$v_H(\sigma)x_H(\sigma) - t_H(\sigma) = v_H(\sigma)x_L(\sigma) - t_L(\sigma) = (\text{IC for H binds })$$

$$\int_)^1 [v_H(\sigma) - v_L(\sigma)](x_H(\sigma) - x_L(\sigma))\, d\sigma \geq 0$$

Last constraint represents IC for poor. We got this from adding IC of rich and poor and since IC of rich holds with equality, this is equivalent to IC for poor. From the first two constraints, substituting values of $t_L(\sigma)$ and $t_H(\sigma)$ gives the following maximization problem of seller.

$$\max_x \int_0^1 [pv_H(\sigma)x_H(\sigma) + (v_L(\sigma) - pv_H(\sigma))\, x_L(\sigma)]\, d\sigma \text{ subject to (1)}$$

Part a.iv Note that coefficient of $x_H(\sigma)$ in objective is always positive while that of $x_L(\sigma)$ is positive iff $v_L(\sigma) - pv_H(\sigma)$ is positive. So, optimal x is given by

$$x_H(\sigma) = 1 \forall \sigma$$
$$x_L(\sigma) = \begin{cases} 0, & \text{if } \frac{v_L(\sigma)}{v_H(\sigma)} \leq p \\ 1, & \text{otherwise} \end{cases}$$

This satisfies the constraint as $x_H(\sigma) - x_L(\sigma)$ is always non-negative. So, the left-hand side of constraint is non-negative. Then from IC of H and IR of L holding with equality, we get the transfers.

$$t_L(\sigma) = \begin{cases} 0, & \text{if } \frac{v_L(\sigma)}{v_H(\sigma)} \leq p \\ v_L(\sigma), & \text{otherwise} \end{cases}$$
$$t_H(\sigma) = \begin{cases} v_H(\sigma), & \text{if } \frac{v_L(\sigma)}{v_H(\sigma)} \leq p \\ v_L(\sigma), & \text{otherwise} \end{cases}$$

b) i) Note that $x_H(\sigma) = 1$ in part a. So, it satisfies the required condition of being non-decreasing in σ^2. For, x to be incentive compatible, we need to show $x_L(\sigma)$ being non-decreasing in σ is a necessary condition. By incentive compatibility of each one revealing true σ, for any $\sigma' < \sigma$

$$v_L(\sigma)x_L(\sigma) - t_L(\sigma) \geq v_L(\sigma)x_L(\sigma') - t_L(\sigma')$$
$$v_L(\sigma')x_L(\sigma') - t_L(\sigma') \geq v_L(\sigma')x_L(\sigma) - t_L(\sigma)$$

Adding both gives

$$[v_L(\sigma') - v_L(\sigma)](x_L(\sigma') - x_L(\sigma)) \geq 0$$

since, v_L is increasing, for above condition to hold, we need

$$x_L(\sigma') - x_L(\sigma) \leq 0 \Leftrightarrow x_L(\sigma') \leq x_L(\sigma)$$

Hence, incentive compatibility implies non-decreasing $x_\theta(\sigma)$ Part ii. With $\frac{v_L(\sigma)}{v_H(\sigma)}$ strictly increasing in σ, optimal x is given by

$$x_H(\sigma) = 1 \forall \sigma$$
$$x_L(\sigma) = \begin{cases} 0, & \text{if } \sigma < \hat{\sigma} \\ 1, & \text{otherwise} \end{cases}$$
$$\text{where } \hat{\sigma} \text{ satisfies } \frac{v_L(\hat{\sigma})}{v_H(\hat{\sigma})} = p$$

Hence, both x_H and x_L are non-decreasing in σ and satisfy the necessary condition needed to be implementable. Whether they are in fact implementable is ambiguous. Note that all H with any σ and L with $\sigma > \hat{\sigma}$ get the object. So, they all must have the same transfer t otherwise they lie. The L with $\sigma < \hat{\sigma}$ don't get the object. So, their transfer is 0.
For them to tell truth, we need their payoff from lying $v_L(\sigma) - t \leq 0 \forall \sigma < \hat{\sigma}$. Hence, the minimum transfer is $v_L(\hat{\sigma})$.
For this transfer low type with $\sigma < \hat{\sigma}$ prefer to tell truth and get 0 surplus.

If they lie, their surplus is negative. Now people getting the object should have at least 0 surplus. We ensured this for low type as $v_L(\sigma) - v_L(\hat{\sigma}) > 0 \forall \sigma > \hat{\sigma}$.

For all high type to participate, we need $v_H(0) \geq t = v_L(\hat{\sigma})$. If this holds, mechanism is implementable. Otherwise, it is not implementable. Hence, the necessary and sufficient condition for part a allocation to be implementable is

$$\lim_{\sigma \to 0} v_H(\sigma) \geq v_L(\hat{\sigma})$$

iii) If x is not non-decreasing, we showed in part (i) that it is not implementable. Also when $\frac{v_L(\sigma)}{v_H(\sigma)}$ is strictly decreasing in σ, x_L is also decreasing in σ, as it is 1 for $\sigma < \hat{\sigma}$ and 0 after that. So, in that case also, it is not implementable.

Question 7

Consider the following bilateral trade problem. There is one buyer and one seller of a single, indivisible object. Valuations, θ_i, are drawn independently across agents from the bounded interval $\left[\underline{\theta}_i, \bar{\theta}_i\right] \subset \mathbb{R}$ according to the CDF F_i with continuous PDF $f_i > 0$, where θ_b is the benefit to the buyer from obtaining the object and θ_s is the opportunity cost to the seller from parting with it. A trading mechanism is a pair (x, t) such that $x(\theta)$ is the probability that the object is given from the seller to the buyer when the parties report type profile $\theta = (\theta_b, \theta_s)$ and $t(\theta) = (t_b(\theta), t_s(\theta))$ is the money paid by each party.

(a) Define formally incentive compatibility (IC) and individual rationality (IR), bearing in mind that a seller who opts out of the mechanism gets to keep the object and may draws utility from it according to his valuation.

(b) Let $U_i(\theta_i) = E[\theta_i x(\theta) - t_i(\theta)]$. Show that

i. $U_b(\underline{\theta}_b) = E[W_b(\theta_b) x(\theta) - t_b(\theta)]$, where $W_b(\theta_b) = \theta_b - (1 - F_b(\theta_b)) / f_b(\theta_b)$

ii. $U_s(\bar{\theta}_s) - \bar{\theta}_s = E[W_s(\theta_s) x(\theta) - t_s(\theta)]$, where $W_s(\theta_s) = -(\theta_s + F_s(\theta_s) / f_s(\theta_s))$

(c) Show that, if $t_b(\theta) + t_s(\theta) = 0$ for every type profile θ, then

$$U_b(\underline{\theta}_b) + U_s(\bar{\theta}_s) - \bar{\theta}_s = E[(W_b(\theta_b) + W_s(\theta_s)) x(\theta)] \geq 0$$

for any feasible (i.e., IR and IC) mechanism.

(d) Assume that $(\underline{\theta}_b, \bar{\theta}_b) \cap (\underline{\theta}_s, \bar{\theta}_s) \neq \emptyset$. A trading mechanism (x^*, t^*) is ex post efficient if $x^*(\theta) = 1$ whenever $\theta_b > \theta_s$ and $x^*(\theta) = 0$ whenever $\theta_b < \theta_s$, and $t_b(\theta) + t_s(\theta) = 0$ for every θ. Show that (x^*, t^*) is ex post efficient only if

$$U_b(\underline{\theta}_b) + U_s(\bar{\theta}_s) - \bar{\theta}_s < 0$$

(Hint: Replace x^* into the virtual surplus and "knead the dough.")

(e) Discuss the implications for efficient, voluntary bilateral trading mechanisms.

(f) What if each party began by owning half of the object instead?

Solution

a) For buyer, incentive compatibility(IC) and individual rationality(IR) are

$$E\left[\theta_b x\left(\theta_b, \theta_s\right)-t_b\left(\theta_b, \theta_s\right)\right] \geq E\left[\theta_b x\left(\theta_b^{\prime}, \theta_s\right)-t_b\left(\theta_b^{\prime}, \theta_s\right)\right] \text{(IC)}$$
$$E\left[\theta_b x\left(\theta_b, \theta_s\right)\ -t_b\left(\theta_b, \theta_s\right)\right] \geq 0 \text{(IR)}$$

IC means telling truth gives highest expected net payoff and IR means this expected net payoff is at least 0. For buyer, incentive compatibility(IC) and individual rationality(IR) are

$$E\left[-\theta_s x\left(\theta_b, \theta_s\right)-t_s\left(\theta_b, \theta_s\right)\right] \geq E\left[-\theta_s x\left(\theta_b, \theta_s^{\prime}\right)-t_s\left(\theta_b, \theta_s^{\prime}\right)\right] \text{(IC)}$$
$$E\left[-\theta_s x\left(\theta_b, \theta_s\right)-t_s\left(\theta_b, \theta_s\right)\right] \geq 0 \text{(IR)}$$

IC means telling truth gives highest expected net payoff which is expected value loss from transaction $-\theta_s x\left(\theta_b, \theta_s\right)$ plus monetary loss $-t_s\left(\theta_b, \theta_s\right)$ and IR means this expected net payoff is at least 0 .

b) i)

$$U_b\left(\theta_b\right)=E\left[\theta_b x\left(\theta_b, \theta_s\right)-t_b\left(\theta_b, \theta_s\right)\right] \equiv \theta_b \bar{x}\left(\theta_b\right)-\bar{t}\left(\theta_B\right)$$

where the bar represent expectation over θ_s given θ_b. This is a standard mechanism which satisfies(from Q2(a) or material covered in class)

$$U_{(b)}\left(\theta_b\right)=U_b\left(\underline{\theta}_b\right)+\int_{\underline{\theta}_b}^{\bar{\theta}_b} \bar{x}(k) d k$$
$$\Rightarrow \theta_b \bar{x}\left(\theta_b\right)-\bar{t}\left(\theta_b\right)=U_b\left(\underline{\theta}_b\right)+\int_{\theta_b}^{\theta_b} \bar{x}(k) d k$$

above equation holds for any θ_b. Taking expectations over θ_b

$$\Rightarrow U_b\left(\underline{\theta}_b\right)=E_{\theta_b}\left[\theta_b \bar{x}\left(\theta_b\right)-\bar{t}\left(\theta_b\right)-\int_{\underline{\theta}_b}^{\theta_b} \bar{x}(k) d k\right]$$

$=E_\theta\left[\theta_b x(\theta)-t(\theta)\right]-\int_{\underline{\theta},}^{\bar{\theta}_b} \int_{\underline{\theta}_b}^{\theta_b} \int_{\underline{\theta}_s}^{\bar{\theta}_s} x\left(k, \theta_s\right) f_s\left(\theta_s\right) d \theta_s d k f_b\left(\theta_b\right) d \theta_b$ (As bar is expectation over θ_s) $= E_\theta\left[\theta_b x(\theta)-t(\theta\right.$ $\int_{\underline{\theta}_s}^{\tilde{\theta}_s} \int_{\underline{\theta}_b}^{\bar{\theta}_b} \int_{\underline{\theta}_b}^{\theta_b} x\left(k, \theta_s\right) d k f_b\left(\theta_b\right) d \theta_b f_s\left(\theta_s\right) d \theta_s$(Reversing order of integrals) By kneading the dough [3],

$$\int_{\underline{\theta}_b}^{\tilde{\theta}_b} \int_{\underline{\theta}_b}^{\theta_b} x\left(k, \theta_s\right) d k f_b\left(\theta_b\right) d \theta_b=\int_{\underline{\theta}_b}^{\bar{\theta}_b} x\left(\theta_b, \theta_s\right)\left(1-F_b\left(\theta_b\right)\right) d \theta_b$$

Putting this into the expression above,

$$\Rightarrow U_b\left(\underline{\theta}_b\right)=E_\theta\left[\theta_b x(\theta)-t(\theta)\right]-\int_{\underline{\theta}}^{\bar{\theta}_s} \int_{\underline{\theta}_b}^{\bar{\theta}_b} x\left(\theta_b, \theta_s\right)\left(1-F_b\left(\theta_b\right)\right) d \theta_b f_s\left(\theta_s\right) d \theta_s$$
$$\Rightarrow U_b\left(\underline{\theta}_b\right)=E_\theta\left[\theta_b x(\theta)-t(\theta)\right]-E_\theta\left[\frac{x\left(\theta_b, \theta_s\right)\left(1-F_b\left(\theta_b\right)\right)}{f_b\left(\theta_b\right)}\right]$$
$$=E\left[\left(\theta_b-\frac{1-F_b\left(\theta_b\right)}{f_b\left(\theta_b\right)}\right) x(\theta)-t(\theta)\right]=E\left[W_b\left(\theta_b\right) x(\theta)-t(\theta)\right]$$

b) ii) Proof:- Define $\hat{U}_s\left(\theta_s\right)=E\left[-\theta_s x\left(\theta_b, \theta_s\right)-t_s\left(\theta_b, \theta_s\right)\right]=-\theta_s \bar{x}\left(\theta_s\right)-\bar{t}\left(\theta_b\right)$ where again bar is expectation over θ_b for a given θ_s. It is the net payoff a seller which must be positive by individual rationality. Here $(-\bar{x}, \bar{t})$ is the mechanism and like in part (i),

$$\hat{U}_s\left(\bar{\theta}_s\right)=\hat{U}_s\left(\theta_s\right)-\int_{\theta_s}^{\bar{\theta}_x} \bar{x}(k) d k$$

This holds for any θ_s. Taking expectation over θ_s

$$\hat{U}_s\left(\bar{\theta}_s\right) = E_\theta\left[-\theta_s x(\theta) - t(\theta)\right] - \int_{\underline{\theta_s}}^{\bar{\theta}_s} \int_{\theta_s}^{\bar{\theta}_\mathbf{s}} \bar{x}(k) dk f\left(\theta_s\right) d\theta_s$$

Again kneading the dough [4],

$$\int_{\underline{\theta_s}}^{\bar{\theta}_\mathbf{s}} \int_{\theta_\mathbf{s}}^{\bar{\theta}_\mathbf{s}} \bar{x}(k) dk f\left(\theta_s\right) d\theta_s = \int_{\underline{\theta_s}}^{\bar{\theta}_\mathbf{s}} \bar{x}\left(\theta_s\right) F_s\left(\theta_s\right) d\theta_s = E_{\theta_\mathbf{s}}\left[\bar{x}\left(\theta_s\right) \frac{F_s\left(\theta_s\right)}{f_s\left(\theta_s\right)}\right] = E_\theta\left[x\left(\theta_s\right) \frac{F_s\left(\theta_s\right)}{f_s\left(\theta_s\right)}\right]$$

First step was reversing the order of integrals. Second step was multiplying and dividing by $f_s\left(\theta_s\right)$ to get into expectations form and last step was rewriting bar as expectation over θ_b. Putting this value in expression above,

$$\hat{U}_s\left(\bar{\theta}_s\right) = E_\theta\left[-\left(\theta_s + \frac{F_s\left(\theta_s\right)}{f_s\left(\theta_s\right)}\right) x(\theta) - t(\theta)\right] = E\left[W_s\left(\theta_s\right) x(\theta) - t_s(\theta)\right]$$

Moreover, $\hat{U}\left(\theta_s\right) = U\left(\theta_s\right) - \theta_s$. So, we have proved the given condition.
(c) WTS that, if $t_b(\theta) + t_s(\theta) = 0$ for every type profile θ, then

$$U_b\left(\underline{\theta}_b\right) + U_s\left(\bar{\theta}_s\right) - \bar{\theta}_s = E\left[\left(W_b\left(\theta_b\right) + W_s\left(\theta_s\right)\right) x(\theta)\right] \geq 0$$

for any feasible (i.e., IR and IC) mechanism. The first half just follows from adding up the equations in b(i) and b(ii)

$$U_b\left(\underline{\theta}_b\right) + U_s\left(\bar{\theta}_s\right) - \bar{\theta}_s = E\left[\left(W_b\left(\theta_b\right) + W_s\left(\theta_s\right)\right) x(\theta)\right] - E\left[t_b(\theta) + t_s(\theta)\right] = E\left[\left(W_b\left(\theta_b\right) + W_s\left(\theta_s\right)\right) x(\theta)\right]$$

since sum of transfers is always 0. It being non-negative follows from individual rationality as the left side is non-negative by individual rationality.

d)

$$U_b\left(\underline{\theta}_b\right) + U_s\left(\bar{\theta}_s\right) - \bar{\theta}_s = E\left[\left(W_b\left(\theta_b\right) + W_s\left(\theta_s\right)\right) x(\theta)\right]$$

Now, for a ex post efficient allocation,

$$E\left[W_b\left(\theta_b\right) x(\theta)\right] = \int_{\underline{\theta},}^{\bar{\theta}_b} W_b\left(\theta_b\right)\left(\int_{\underline{\theta}_s}^{\theta_b} f_s\left(\theta_s\right) d\theta_s\right) f_b\left(\theta_b\right) d\theta_b = \int_{\underline{\theta}}^{\bar{\theta}_b} W_b\left(\theta_b\right) F_s\left(\theta_b\right) f_b\left(\theta_b\right) d\theta_b$$

Above step is just limiting the expectation in the range when $x(\theta) = 1$, that is, $\theta_b > \theta_s$, that is $\theta_b \in \left[\underline{\theta}_s, \bar{\theta}_b\right]$ and $\theta_s \in \left[\underline{\theta}_s, \theta_b\right]$. Substituting expression for $W_b\left(\theta_b\right)$

$$E\left[W_b\left(\theta_b\right) x(\theta)\right] = \int_{\underline{\theta}_-}^{\bar{\theta}_b} \left(\theta_b f_b\left(\theta_b\right) - \left(1 - F_b\left(\theta_b\right)\right)\right) F_s\left(\theta_b\right) d\theta_b$$

Similarly,

$$\begin{aligned} E\left[W_s\left(\theta_s\right) x(\theta)\right] &= -\textstyle\int_{\underline{\theta}}^{\bar{\theta}_b} \left(\theta_s f_s\left(\theta_s\right) + F_s\left(\theta_s\right)\right)\left(1 - F_b\left(\theta_s\right)\right) d\theta_s \\ &= -\textstyle\int_{\underline{\theta}_s}^{\bar{\theta}_b} \theta_s f_s\left(\theta_s\right)\left(1 - F_b\left(\theta_s\right)\right) d\theta_s - \int_{\underline{\theta}_s}^{\bar{\theta}_b} F_s\left(\theta_s\right)\left(1 - F_b\left(\theta_s\right)\right) d\theta_s \end{aligned}$$

Integrating first term by parts, it becomes

$$E\left[W_s\left(\theta_s\right)x(\theta)\right] = -\left[\theta_s\left(1-F_b\left(\theta_s\right)\right)F_s\left(\theta_s\right)\right]_{\underline{\theta}_s}^{\bar{\theta}_s} + \int_{\underline{\theta}}^{\bar{\theta}_b} F_s\left(\theta_s\right)\left(1-F_b\left(\theta_s\right)\right)d\theta_s -$$

$$\int_{\underline{\theta}_s}^{\bar{\theta}_b} \theta_s f_b\left(\theta_s\right)F_s\left(\theta_s\right)d\theta_s - \int_{\underline{\theta}_s}^{\bar{\theta}_b} F_s\left(\theta_s\right)\left(1-F_b\left(\theta_s\right)\right)d\theta_s$$

$$\Rightarrow E\left[W_s\left(\theta_s\right)x(\theta)\right] = 0 - \int_{\underline{\theta}_s}^{\tilde{\theta}_s} \theta_s f_b\left(\theta_s\right)F_s\left(\theta_s\right)d\theta_s$$

Summing up both expressions,

$$E\left[\left(W_b\left(\theta_b\right)+W_s\left(\theta_s\right)\right)x(\theta)\right] = \int_{\underline{\theta}}^{\bar{\theta}_b}\left(\theta_b f_b\left(\theta_b\right)-\left(1-F_b\left(\theta_b\right)\right)\right)F_s\left(\theta_b\right)d\theta_b - \int_{\underline{\theta}}^{\bar{\theta}_b}\theta_s f_b\left(\theta_s\right)F_s\left(\theta_s\right)d\theta_s$$

$$\Rightarrow E\left[\left(W_b\left(\theta_b\right)+W_s\left(\theta_s\right)\right)x(\theta)\right] = \int_{\underline{\theta}_s}^{\bar{\theta}_b} -\left(1-F_b\left(\theta_b\right)F_s\left(\theta_b\right)d\theta_b < 0\right.$$

as cdf is always between 0 and 1 and the interval is non-empty by assumption.

(e) Discuss the implications for efficient, voluntary bilateral trading mechanisms.

(f) What if each party began by owning half of the object instead?

From parts c and d, there is a contradiction for any IR,IC,ex post and zero net-transfers mechanism. So, for any efficient, voluntary bilateral trading, some money must be burnt(net transfers must be negative) to ensure truthtelling. We can not have all desirable properties-participation, incentive compatibility, efficiency and no monetary loss. Moreover, the assumption in part f does not affect the result.It just changes the valuation of object for each side in the trade. Say, buyer has $\frac{\theta_b}{2}$ and seller has $\frac{\theta_x}{2}$ and trade is happening over rest half.