Recitation 4

[Definitions used today]

- Topkis theorem, Supermodularity, Increasing Differences

Question 1 [Midterm]

Suppose that a firm with production function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ such that $f(0) = 0$ chooses its production plan $(x; z)$ at prices $w \in \mathbb{R}^n_{++}$ of inputs and $q \in \mathbb{R}_{++}$ of the output in such a way that minimizes the cost of producing z at prices w, and the marginal cost $\frac{\partial C^*}{\partial z}(w; z)$ equals the output price q:

a Under what conditions on f is the firm maximizing its production? Be as general as you can. Prove you answer.

b Suppose that cost function C^* is strictly concave in$z$. Show that the firm makes a loss (strictly negative profit) when following the marginal cost rule whenever the output is non-zero.

Question 2 [Topkis theorem]

If S is a lattice, f is supermodular in x, and f has nondecreasing differences in $(x; t)$, then $\varphi^*(t) = \arg\max_{x \in S} f(x, t)$ is monotone nondecreasing in t.

Question 3 [Midterm 2017] or ∼ 82,89 [II.1 Spring 2009 majors]

Consider a profit maximizing firm with single output and n inputs, with production function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ assumed strictly increasing, continuous (but possibly nondifferentiable), and $f(0) = 0$. Let $q \in \mathbb{R}_{++}$ be the price of output and $w \in \mathbb{R}^n_{++}$ be the vector of prices of inputs. The firm's profit maximization problem is

$$\max_{x \geq 0} [qf(x) - wx]$$

a Show that if the production function f is supermodular, then the firm's input demand x is monotone non-increasing in input prices, that is if $w \leq w'$ for $w, w \in \mathbb{R}^N_{++}$ then $x(w, q) \geq x(w, q)$. You may assume that input demand x is single valued. Production function is strictly increasing but need not be differentiable.

b Under what conditions on f is the solution $x(w, q)$ unique? Be as general as you can and prove your answer

c Give an example of strictly increasing function that is not supermodular.

Question 4

Consider a $C \subset \mathbb{R}^L$, $T \subset \mathbb{R}$. Define function F in following way:

$$F : \mathbb{R}^L \times T \to \mathbb{R} \quad F(x, t) = \bar{F}(x) + f(x, t)$$

where $f : \mathbb{R} \times T \to \mathbb{R}$ is supermodular and $\bar{F} : \mathbb{R}^L \to \mathbb{R}$. Assume that:

$$\forall \quad t'' > t' \quad x'' \in \operatorname*{argmax}_{x \in C} F(x, t'') \quad x' \in \operatorname*{argmax}_{x \in C} F(x, t')$$

Show that if $x_i' > x_i''$ then

$$\forall \quad t'' > t' \quad x'' \in \underset{x \in C}{\operatorname{argmax}} F(x, t') \quad x' \in \underset{x \in C}{\operatorname{argmax}} F(x, t'')$$

Question 5

Let $\{f(s,t)\}$ $t \in T$ be a family of density functions on $S \subset R$. T is a poset (partially ordered set). Consider

$$v(x,t) = \int_S u(x,s) f(s,t) ds$$

Prove the following statement. Suppose u has increasing differences and that $\{f(\cdot,t)\}$ $t \in T$ are ordered with t by first order stochastic dominance. Then v has increasing differences in (x,t).

Question 6 Suppose that utility function $u : \mathbb{R}_+^\ell \to \mathbb{R}$ is supermodular, strictly concave, and locally non-satiated. Then the Walrasian demand function $x^*(\cdot)$ is a nondecreasing function of income, i.e.,

$$x^*(p, w') \geq x^*(p, w), \ \forall w' \geq w \geq 0, \ \forall p \gg 0.$$

In other words, the demand for every good is normal.