Recitation 6

[Definitions used today]

- Risk aversion and strict risk aversion, Jensen's inequalities.
- FOSD, SOSD, more risky than, Pratt's theorem

Question 1 [Risk compensation] 107 [I.2 Fall 2010 majors]

Consider an agent with expected utility function $E[v(\cdot)]$, where the von Neumann-Morgenstern utility function v is strictly increasing. Consider risk compensation $\rho(w, t\tilde{z})$ as a function of scale factor t for arbitrary $t \in \mathbb{R}_+$.

a State a definition of risk compensation $\rho(w, \tilde{z})$ for risky gamble $\tilde{z}$ with $E(\tilde{z}) = 0$ at deterministic wealth w

b Show that $\rho(w, t\tilde{z})$ is a strictly increasing function of t that takes zero value at $t = 0$ for every w and $\tilde{z}$ with $E(\tilde{z}) = 0$, if and only if the agent is strictly risk averse.

Question 2 [Pratt] 91 [II.1 Fall 2009 majors]

Consider an agent whose preferences over real-valued random variables (or state-contingent consumption plans) are represented by an expected utility function with strictly increasing and twicedifferentianble vN-M utility $v : \mathbb{R} \to \mathbb{R}$. Let $\rho(w, \tilde{z})$ denote the risk compensation for random variable $\tilde{z}$ with $\mathbb{E}(z) = 0$ at risk-free initial wealth w. Let $A(w)$ denote the Arrow-Pratt measure of risk aversion at w.

a Prove that A is an increasing function of w if and only if risk compensation ρ is an increasing function of w for every $\tilde{z}$ with $\mathbb{E}(\tilde{z}) = 0$ and $\tilde{z} \neq 0$.

b Derive an explicit expression for risk compensation for quadratic utility $v(x) = -(\alpha - x)^2$ where $\alpha > 0$. Prove that this quadratic utility is, up to an increasing linear transformation, the only utility function with risk compensation of the form you derived.

c Give an example of two vN-M utility function v_1 and v_2 such that neither v_1 is more risk averse than v_2, nor v_2 is risk averse than v_1 in the sense of the Theorem of Pratt.

Question 3

There are three states with equal probabilities $\pi_s = \frac{1}{3}$ for $s \in \{1, 2, 3\}$. Consider two state contingent consumption plans $z = (8, 2, 2)$, and $y = (3, 3, 6)$

a Does y FOSD dominate z?

b Is z more risky than y?

Question 4 124 [I.2 Fall 2011 majors]

Consider two real-valued random variables $\tilde{y}$ and $\tilde{z}$ on some state space (i.e. probability space). Let F_y and F_z be their cumulative distribution functions, and $E(\tilde{z})$ and $E(\tilde{y})$ their expected values.

a State a definition of $\tilde{z}$ first-order stochastically dominating (FSOD) $\tilde{y}$. Show that if $\tilde{z}$ FSOD $\tilde{y}$, then $E(\tilde{z}) \geq E(\tilde{y})$

b Show that, if $\tilde{z}$ FSOD $\tilde{y}$ and $E(\tilde{z}) = E(\tilde{y})$, then $\tilde{y}$ and $\tilde{z}$ have the same distribution, i.e., $F_y(t) = F_z(t)$ for every $t \in \mathbb{R}$. If you find it convenient, you may assume in your proof that random variables $\tilde{y}$ and $\tilde{z}$ have densities, or alternatively that $\tilde{y}$ and $\tilde{z}$ are discrete random variables (i.e., take finitely many values).

c State a definition of $\tilde{z}$ second-order stochastically dominating (SSOD) $\tilde{y}$. Show that if $\tilde{z}$ ssn $\tilde{y}$, then $E(\tilde{z}) \geq E(\tilde{y})$

d Show that if $\tilde{z}$ FSOD $\tilde{y}$, then $\tilde{z}$ SSD $\tilde{y}$.

e State a definition of $\tilde{y}$ being more risky than $\tilde{z}$. Give a brief justification for why it is a sensible definition of more risky.

Question 5 [Stochastic Dominance and Risk]

Consider two real-valued random variables y and z on some finite state space with $\mathbb{E}[y] = \mathbb{E}[z]$.

a Prove that if y is more risky than z, then $\mathbb{E}[v(z)] \geq \mathbb{E}[v(y)]$ for every nondecreasing continuous and concave function $v : \mathbb{R} \to \mathbb{R}$. You may assume v is twice differentiable.

b Give an example of two random variables y and z such that $y \neq z, \mathbb{E}[y] = \mathbb{E}[z]$ and neither z is more risky than y nor y is more risky than z.

Question 6

Consider an optimal portfolio choice problem with one risky asset with return $\tilde{r}$ and a risk-free asset with return r_f. Suppose that the agent's vNM utility function is $v(x) = -(\alpha - x)^2$ for some $\alpha > 0$. Assume that $\alpha > wr_f$ where $w > 0$ is agent's wealth. Negative investment (i.e. short selling) is permitted for both assets.

a Find the optimal investment in the risky asset as a function of expected return and the variance of the risky return.

b Suppose that the return $\tilde{r}$ on the risky asset is changed to a more risky return $\tilde{r}'$ with the same expectation $\mathbb{E}[\tilde{r}'] = \mathbb{E}[\tilde{r}]$. Assume $\mathbb{E}[\tilde{r}] > r_f$. Prove that the optimal investment in the risky asset with more risky return $\tilde{r}'$ is smaller than the optimal investment with return $\tilde{r}$, all else unchanged.